Eq. (1.4) with the boundary conditions

$$u|_{\psi=-V(0,x)} = f_\varepsilon(x,\psi), \quad u|_{x=0} = g_\varepsilon(\psi), \quad u(x,\psi) \xrightarrow[\psi\to\infty]{} U(x) \tag{3.3}$$

Here $f_\varepsilon > 0$, $g_\varepsilon > 0$; $f_\varepsilon(x,\psi) \to 0$, $g_\varepsilon(\psi) \to u_*(\psi)$ for $\varepsilon \to 0$. In addition, f_ε, g_ε are smooth functions and the consistency condition is satisfied at the point (0, 0). In the same way we obtain a positive solution $u_b(x,\psi)$ of the problem (1.5),(1.6) in G_a for some a as the limit of solutions $u_b^\varepsilon(x,\psi)$ of the problem (1.5),(3.3) as $\varepsilon \to 0$.

The function $s = u_b^\varepsilon - u_p^\varepsilon$ satisfies the linear equation

$$\frac{\partial s}{\partial x} - 2\left(\frac{\partial u_b^\varepsilon}{\partial\psi} + \frac{\partial u_p^\varepsilon}{\partial\psi}\right)\frac{\partial s}{\partial\psi} = (u_b^\varepsilon + u_p^\varepsilon)\frac{\partial^2 s}{\partial\psi^2} - \left(\frac{\partial^2 u_b^\varepsilon}{\partial\psi^2} + \frac{\partial^2 u_p^\varepsilon}{\partial\psi^2}\right)s + \frac{1}{u_p^\varepsilon}\frac{dp}{dx}$$

Since $s = 0$ on the boundary of the domain G_a, $dp/dx \geqslant 0$, $u_b^\varepsilon > 0$, $u_p^\varepsilon > 0$, and the second derivatives of the functions u_b^ε and u_p^ε are bounded with respect to ψ, it then follows from the maximum principle that $u_p^\varepsilon \leqslant u_b^\varepsilon$ in G_a. By a limiting transition we obtain the inequality $u_p(x,\psi) \leqslant u_b(x,\psi)$ in G_a From this it follows that when $dp/dx \geqslant 0$ boundary layer separation takes place in D_a for $a = x_2$ if separation takes place in D_a for $a = x_2$, when $dp/dx \equiv 0$. This completes the proof of the theorem.

Corollary. If $dp/dx \geqslant 0$ and $v_0(x) = m = \text{const} > 0$, boundary layer separation takes place in D_a for some a.

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ON SINGLE-VALUED SOLUTION OF THE FUNDAMENTAL BOUNDARY-VALUE PROBLEM IN THE THEORY OF THE THERMAL BOUNDARY LAYER

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We consider the system of thermal boundary layer equations for a two-dimensional steady forced-convective flow of an incompressible fluid. Our principal object of study being the equation for the temperature. We prove the single-valued solvability of the fundamental boundary-value problem for this equation.

The problem of the single-valued solvability of the fundamental problems of dynamic boundary layer theory for the steady flow of an incompressible fluid was studied in [1 − 5]. The temperature equation was studied in [6] in connection with the case involving solution of the dynamic system of equations relative to the problem of extending the boundary layer [1 − 5].

1. Consider the system of equations of the thermal boundary layer for the forced-convective flow of an incompressible fluid (see [7])

$$\nu u_{yy} - u u_x - v u_y = - U U_x, \quad u_x + v_y = 0 \tag{1.1}$$

$$a T_{yy} - u T_x - v T_y = f(x, y) = -\nu / c\,(u_y)^2 \tag{1.2}$$

Here ν, a and c are known positive constants and $U(x)$ is the specified longitudinal velocity component of the outer flow. Initially we determine the functions $u(x, y)$ and $v(x, y)$ from the dynamic equations (1.1) and the boundary conditions corresponding to them; we then use the result to determine the temperature $T(x, y)$ in the boundary layer from Eqs. (1.2) and the associated boundary conditions, which we give below.

Let $u(x, y)$ and $v(x, y)$ be the solution in the domain $Q\,\{0 < x < X,\ 0 < y < \infty\}$ of Eqs. (1.1) with the following boundary conditions:

$$u|_{y=0} = 0, \quad u|_{x=0} = 0, \quad v|_{y=0} = v_0(x), \quad \lim_{y\to\infty} u(x, y) = U(x) \tag{1.3}$$

Here $v_0(x)$ and $U(x)$ are specified smooth functions, $U(0) = 0$, $U(x) > 0$ for $x > 0$ and $U'(0) > 0$.

Uniqueness and existence theorems were established for the solution of the problem (1.1), (1.3) in [3, 4]. We shall not give these theorems here but merely note those properties of the functions u and v which are used to study Eq. (1.2) in the strip Q; for brevity we call them the Conditions A: in any arbitrary compactum lying in the strip Q the functions $u(x, y)$, $v(x, y)$ and $u_y(x, y)$ satisfy Hölder condition; $u(x, y) > 0$ for x, $y > 0$; $u(x, 0) = 0$, $u(0, y) = 0$; $u(x, y) \leqslant U(x)$ everywhere in Q, where the function $U(x)$ is continuous for $x \geqslant 0$ and

$$\int_0^X \frac{dt}{U(t)} = +\infty \tag{1.4}$$

the function $v(x, y)$ is bounded for bounded y, and $u_y(x, y) \to 0$ for $y \to \infty$. It is evident that Eq. (1.2) in the domain Q is a parabolic equation which is degenrate for $x = 0$ and $y = 0$.

Before formulating a properly-posed boundary-value problem in Q for Eq. (1.2), we make the following notes.

Note 1. It is known (see, for example, [8, 9]) that owing to the conditions (1.3) and (1.4) no boundary condition can be specified for $T(x, y)$ at $x = 0$. We show that a bounded solution of an equation of the form (1.2) with the boundary condition

$$T|_{y=0} = T_0(x) \tag{1.5}$$

where $T_0(x)$ is a specified continuous function on $0 \leqslant x \leqslant X$, may prove to be non-unique. In fact, the function

$$T(y) = \int_0^y \exp\left\{\frac{\alpha}{2} t^2\right\} dt$$

which is bounded and nonzero in the strip Q, satisfies Eq. (1.2) for $a = 1$, $u = x^k$, $v = \alpha y$, $f = 0$ ($k \geqslant 1$, $\alpha = \text{const} < 0$) and also the homogeneous boundary condition $T|_{y=0} = 0$. It follows from this that for the conditions considered the solution of the problem (1.2), (1.5) is not unique. Thus, in addition to the condition (1.5), yet another condition must be given. In boundary layer theory a condition which serves the puropse is $\lim_{y\to\infty} T(x, y) = T_\infty$, where T_∞ is a specified constant.

Theorem . Suppose that the coefficients and the right side of Eq. (1.2) satisfy the Conditions A and, in addition, that

$$\int_0^{+\infty} \exp\left\{\int_0^t \beta(\tau)\,d\tau\right\} dt < \infty, \qquad \beta(y) = \max_{0\leqslant x\leqslant X} v(x, y) \tag{1.6}$$

$$|f(x, y)| \leqslant g(y) \tag{1.7}$$

where the positive function $g(y) \to 0$ for $y \to \infty$ in such a way that

$$\int_0^{+\infty} g(\tau) \exp\left\{-\int_0^\tau \beta(s)\,ds\right\} d\tau < \infty \tag{1.8}$$

Then in Q Eq. (1.2) has a unique bounded solution satisfying the conditions

$$T|_{y=0} = T_w(x), \quad \lim_{y\to\infty} T(x, y) = T_\infty \tag{1.9}$$

where $T_w(x)$ is a specified continuous function (temperature of the wall of the body over which the flow takes place), $T_w'(x)$ is bounded, and T_∞ is a specified constant (temperature of the outer flow).

2. For proving the theorem we find it convenient to make a change in the independent variables, namely,

$$x = x, \qquad \eta = 1 - \frac{1}{1+y} \tag{2.1}$$

for which the strip Q goes over into the rectangle $D\{0 < x < X, 0 < \eta < 1\}$, and Eq. (1.2), wherein with no loss in generality, we can assume $a = 1$ and replace it by the following equation in the domain D:

$$L(T) \equiv (1-\eta)^4 T_{\eta\eta} - u(x, \eta) T_x + b(x, \eta) T_\eta = f(x, \eta) \tag{2.2}$$

$$b(x, \eta) = -(1-\eta)^2 [2(1-\eta) + v(x, \eta)]$$

The conditions (1.9) become the boundary conditions

$$T|_{\eta=0} = T_w(x), \qquad T|_{\eta=1} = T_\infty \tag{2.3}$$

We prove the existence of a solution of the problem (2.2), (2.3). For this purpose we consider Eq. (2.2) in the rectangle

$$D_\delta\{\delta < x \leqslant X, \quad \delta < \eta < 1-\delta\}, \quad 0 < \delta < 1/4$$

with the following conditions on the boundary of D_δ:

$$T|_{\eta=\delta} = T_w(x), \quad T|_{\eta=1-\delta} = T_\infty, \quad T|_{x=\delta} = T_1^\delta(\eta) \tag{2.4}$$

where the function $T_1^\delta(\eta)$ is chosen to satisfy the conditions

$$T_1^\delta(\delta) \equiv T_w(\delta) \quad \text{for} \quad \eta \leqslant 1/3, \qquad T_1^\delta(\eta) \equiv T_\infty \quad \text{for} \quad \eta > 2/3$$

$$|T_1^\delta(\eta)| \leqslant \max\{|T_\infty|, \max_{0\leqslant x\leqslant X} |T_w(x)|\} \equiv M$$

A solution $T^\delta(x,\eta)$ of the problem (2.2), (2.4) exists, and by the maximum principle for nondegenerate parabolic equations (see, for example, [10])

$$|T^\delta(x,\eta)| \leqslant M \tag{2.5}$$

where M does not depend on δ. By virtue of Schauder-type estimates [10], there exist in an arbitrary fixed rectangle D_δ, estimates for the Hölder norms of the solution T^δ and its derivatives T^δ_η, $T^\delta_{\eta\eta}$, T^δ_x which are uniform with respect to δ.

Based on these estimates, we use the diagonal process to select a subsequence T^{δ_m} ($m = 1, 2, 3, \ldots$) which, together with the derivatives appearing in Eq. (2.2), converges for $m \to \infty$ ($\delta_m \to 0$ for $m \to \infty$) in every closed domain lying strictly inside D. Letting $m \to \infty$ in the equation for T^{δ_m}, we find that the limit function $T(x,\eta)$ satisfies Eq. (2.2) in the rectangle D.

To prove that the condition $T(x,0) = T_w(x)$ is satisfied we estimate the difference $T^\delta(x,\eta) - T_w(x) = S^\delta(x,\eta)$ for small η. The equation

$$L(S^\delta) = f(x,\eta) + u(x,\eta)\; T_w'(x)$$

which is satisfied by the function $S^\delta(x,\eta)$, is now considered in the domain D_δ' $\{\delta \leqslant x \leqslant X,\ \delta < \eta < {}^1/_3\}$. Let $|f(x,\eta)| \leqslant M_1$, and $u(x,\eta)\,|T_w'(x)| \leqslant M_2$ in the domain D_δ'. We introduce the auxiliary function $Y(\eta) = K(1 - e^{-N\eta})$. We choose the constant $N > 0$ from the condition $N(1-\eta)^4 \geqslant |b(x,\eta)| + 1$. This is possible since the coefficient $v(x,y)$ is bounded for bounded y, or, in accord with the substitution (2.1), for $\eta \leqslant \eta_0 < 1$. We choose the constant $K > 0$ so that

$$K \geqslant \max\left\{\frac{2M}{1-e^{-N/3}}\ \frac{M_1+M_2}{Ne^{-N}}\right\} \tag{2.6}$$

where M is the quantity appearing in the inequality (2.5). Computing $L(Y)$, by virtue of the choice of N, we have

$$L(Y) = -KNe^{-N\eta}(N(1-\eta)^4 - b(x,\eta)) \leqslant -KNe^{-N} < 0$$

Consider the function $Y \pm S^\delta(x,\eta)$. By virtue of inequality (2.6), we have in D_δ'

$$L(Y \pm S^\delta) \leqslant -KNe^{-N} \pm f(x,\eta) \pm u(x,\eta)\quad T_w'(x) \leqslant -KNe^{-N} + M_1 + M_2 \leqslant 0$$

From the relations (2.4) – (2.6) it follows that the function $Y \pm S^\delta \geqslant 0$ on the boundary of the domain D_δ' lying on the lines $x = \delta$, $\eta = \delta$, $\eta = {}^1/_3$. From this it follows by the maximum principle that $Y \pm S^\delta \geqslant 0$ everywhere in D_δ'. Hence we have the estimate $|S^\delta(x,\eta)| \leqslant Y(\eta)$ which is uniform with respect to δ and x. Then, letting $\delta \to 0$ and $\eta \to 0$, we obtain $T(x,0) = T_w(x)$.

We now prove that $T(x,\eta)$ also satisfies the second of the conditions (2.3). To do this, we estimate $F^\delta(x,\eta) = T^\delta(x,\eta) - T_\infty$ for small $1-\eta$. We consider the equation $L(F^\delta) = f(x,\eta)$, which the function $F^\delta(x,\eta)$ satisfies, in the domain D_δ'' $\{\delta < x \leqslant X,\ {}^2/_3 \leqslant \eta < 1-\delta\}$. Consider the function

$$Z(\eta) = K_1 \int_\eta^1 G(t)\, G_1(t)\, dt \tag{2.7}$$

where

$$G(t) = \frac{1}{(1-t)^2} \exp\left\{\int_0^t \frac{\beta(\tau)\, d\tau}{(1-\tau)^2}\right\}, \qquad G_1(t) = 1 + \int_0^t \frac{g(\tau)\, d\tau}{(1-\tau)^4\, G(\tau)}$$

the functions $\beta(\tau)$ and $g(\tau)$ being the same as in the statement of the theorem. The integral (2.7) exists by virtue of the inequalities (1.6) and (1.8) and defines a positive function for $\eta < 1$. We choose the constant $K_1 \geqslant 1$ from the condition $Z\,(^2/_3) \geqslant 2M$ where M is the constant of inequality (2.5). We have

$$L\,(Z) = -\,K_1 G_1\,(\eta)\,[(1-\eta)^4\,G'\,(\eta) + b\,(x,\eta)\,G(\eta)] - K_1\,g\,(\eta) \tag{2.8}$$

Since

$$(1-\eta)^4 G'\,(\eta) + b\,(x,\eta)\,G\,(\eta) = \exp\left\{\int_0^{\eta} \frac{\beta\,(\tau)\,d\tau}{(1-\tau)^2}\right\}[\beta\,(\eta) - v\,(x,\eta)] \geqslant 0$$

from the definition of the function $\beta\,(\eta)$, we obtain from Eq. (2.8)

$$L\,(Z) \leqslant -\,K_1\,g\,(\eta)$$

It is easy to see that the function $Z\,(\eta) \pm F^{\delta}\,(x,\eta) \geqslant 0$ on the boundary of D_{δ}'' lying on the lines $x = \delta$, $\eta = {}^2/_3$, $\eta = 1-\delta$. By virtue of the choice of $K_1 \geqslant 1$, inside D_{δ}'' we have $L\,(Z \pm F^{\delta}) \leqslant -\,K_1\,g\,(\eta) \pm f\,(x,\,\eta) \leqslant -\,K_1 g\,(\eta) + g\,(\eta) \leqslant 0$. From this we find by the maximum principle that $Z \pm F^{\delta} \geqslant 0$ everywhere in the domain D_{δ}'', or, $|\,F^{\delta}\,(x,\eta)\,| \leqslant Z\,(\eta)$ in this domain. Letting $\delta \to 0$ in the last inequality, we obtain that $T\,(x,\eta) \to T_{\infty}$ for $\eta \to 1$, uniformly with respect to x, QED.

It is obvious that if $f \equiv 0$ in Eq. (1.2) (which is the situation when heat generated by friction is neglected), the condition (1.7) is eliminated and for the function $Z\,(\eta)$ we can take

$$K_1 \int_{\eta}^{1} G\,(t)\,dt$$

We proceed now to prove the uniqueness of the solution of the problem (2.2), (2.3). Suppose that two solutions of the problem exist and consider their difference

$$T\,(x),\eta) = T_1\,(x,\eta) - T_2\,(x,\eta)$$

For an arbitrary $\varepsilon > 0$ we can find a $\delta\,(\varepsilon)$ such that for $\eta = 1 - \delta\,(\varepsilon)$ we have $|\,T\,(x,\eta)\,| \leqslant \varepsilon$. The function $V = T\,(x,\eta) - \varepsilon$ satisfies the equation $L\,(V) = 0$, moreover, $V \leqslant 0$ for $\eta = 0$ and $\eta = 1 - \delta\,(\varepsilon)$. We show that $V \leqslant 0$ everywhere in the rectangle $D_{\varepsilon}\,\{0 < x \leqslant X,\ 0 < \eta < 1 - \delta\,(\varepsilon)\}$. In the equation $L\,(V) = 0$ we make the substitution

$$V\,(x,\eta) = H\,(\eta)\,R\,(x,\eta),\ H\,(\eta) > 0$$

then for the function $R\,(x,\eta)$ in D_{ε} we obtain the equation

$$L\,(R) \equiv (1-\eta)^4\,R_{\eta\eta} - uR_x + BR_{\eta} + CR = 0 \tag{2.9}$$

$$B\,(x,\eta) = \frac{2\,(1-\eta)^4 H' + bH}{H}, \qquad C\,(x,\eta) = \frac{(1-\eta)^4\,H'' + bH'}{H}$$

We select the function $H\,(\eta) > 0$ so that in Eq. (2.9) we shall have $C\,(x,\eta) \leqslant c_0 = \mathrm{const} < 0$. Consider the function

$$\Phi\,(x) = \int_x^X \frac{dt}{U\,(t)}$$

where $U\,(t)$ is the function that appears in relation (1.4). We have

$$L\,(\Phi) = C\,(x,\eta)\,\Phi\,(x) + \frac{u\,(x,\eta)}{U\,(x)} \leqslant c_0\,\Phi\,(x) + 1.$$

Since $c_0 < 0$ and $\Phi\,(x) \to \infty$ for $x \to 0$, we can find an x_0 such that for all $x \leqslant x_0$ we have $L\,(\Phi) \leqslant 0$. Then for arbitrary $\gamma > 0$ we obtain

$$L\left(\gamma\Phi(x)-R(x,\eta)\right)\leqslant 0, \quad \text{for} \quad x\leqslant x_0$$

For fixed γ we can find $x_1(\gamma)>0$ such that $\gamma\Phi(x_1)-R(x_1,\eta)\geqslant 0$ for $0\leqslant\eta\leqslant 1-\delta(\varepsilon)$. In addition we also have $\gamma\Phi(x)-R(x,\eta)\geqslant 0$ for $\eta=0$ and for $\eta=1-\delta(\varepsilon)$ since $R(x,0)<0$ and $R(x,1-\delta(\varepsilon))\leqslant 0$. Hence, it follows from the maximum principle that $\gamma\Phi(x)-R(x,\eta)\geqslant 0$ everywhere in rectangle $\{x_1\leqslant x\leqslant x_0,\ 0\leqslant\eta\leqslant 1-\delta(\varepsilon)\}$. Since γ is arbitrary, it follows from the inequality $R(x,\eta)\leqslant\gamma\Phi(x)$ just proved, that $R\leqslant 0$ in the rectangle $\{0\leqslant x\leqslant x_0,\ 0\leqslant\eta\leqslant 1-\delta(\varepsilon)\}$. But then by the maximum principle this is also true in the rectangle $\{x_0\leqslant x\leqslant X,\ 0<\eta<1-\delta(\varepsilon)\}$, since there $L(R)=0$ also and

$$R\mid_{x=x_0}\leqslant 0, \quad R\mid_{\eta=0}\leqslant 0, \quad R\mid_{\eta=1,\ \delta(\varepsilon)}\leqslant 0$$

Thus we have proved that $R\leqslant 0$ everywhere in D_ε. From this it then follows that $T(x,\eta)\leqslant\varepsilon$ in D_ε. By virtue of the symmetry of T_1 and T_2 we have $|T(x,\eta)|\leqslant\varepsilon$ in D_ε. Letting $\varepsilon\to 0$, we obtain $T_1\equiv T_2$. This completes the proof of the theorem.

Note 2. We have shown above that the solution of the problem (1.1), (1.3), constructed in [3, 4], satisfies the Conditions A of the theorem. However, in the theorem, besides the Conditions A, we need also to satisfy the conditions (1.6), (1.7). Proceeding from the results presented in [3], we can show that the solution $u(x,y)$ and $v(x,y)$ of the problem (1.1), (1.3) also satisfies the conditions (1.6) and (1.7).

Note 3. In the formulation of the problem (1.2), (1.9) we assumed that $T_\infty=\text{const}$. We show now that we cannot specify a nonconstant value at infinity for the solution $T(x,y)$ of Eq. (1.2) which is bounded in the strip Q.

Let us assume that the solution $T(x,y)$ of the problem is such that $\lim_{y\to\infty}T(x,y)=T_\infty(x)$. By Eqs. (2.1) this is equivalent to the condition $T(x,\eta)_{\eta=1}=T_\infty(x)$. For an arbitrary $\varepsilon>0$ we can find a $\delta(\varepsilon)$ such that $|T(x,1-\delta)-T_\infty(x)|<\varepsilon$. Let us set $T^\delta(x,\eta)-T_\infty(\delta)=F^\delta(x,\eta)$. Equation (2.2), which the function $F^\delta(x,\eta)$ satisfies, is now considered in the domain $D_\delta''\{\delta<x\leqslant X,\ {}^2/_3\leqslant\eta<1-\delta(\varepsilon)\}$. Further, considering the function $Z(\eta)+\varepsilon\pm F^\delta(x,\eta)$, where $Z(\eta)$ is defined by Eq. (2.7), and repeating word for word all the steps followed in proving that the second of the conditions (1.9) is satisfied, we obtain the following estimate in the domain $\overline{D}_\delta''$:

$$|F^\delta(x,\eta)|\leqslant\varepsilon+Z(\eta)$$

uniformly in δ and x. From this, letting $\delta\to 0$ and noting that $Z(\eta)\to 0$ for $\eta\to 1$, we have $|T(x,1),-T_\infty(0)|\leqslant\varepsilon$. Since ε is arbitrary, we obtain $T(x,1)\equiv T_\infty(0)$, i.e. $T_\infty(x)\equiv T_\infty(0)=\text{const}$, QED.

3. For the study of the system (1.1), (1.2) we can use the Crocco transformation: $\xi=x,\ \eta=u(x,y)/U(x)$. Then the domain Q goes over into a domain D, and instead of the dynamic system of Eqs. (1.1), we obtain in D the following equation for $W=u_y/U(\xi)$:

$$\nu W^2W_{\eta\eta}-\eta U(\xi)W_\xi+(\eta^2-1)U_\xi W_\eta-\eta U_\xi W=0 \tag{3.1}$$

The conditions (1.3) transform into the following conditions for W:

$$W\mid_{\eta=1}=0, \quad (\nu WW_\eta-v_0(\xi)W+U_\xi)\mid_{\eta=0}=0 \tag{3.2}$$

The Crocco transformation changes the temperature equation (1.2) into an equation of the form

$$aW^2 T_{\eta\eta} - \eta U(\xi)\, T_\xi - [(\nu - a)\, WW_\eta + U_\xi (1 - \eta^2)]\, T_\eta = \\ - \nu C^{-1} U^2(\xi)\, W^2(\xi, \eta) \qquad (3.3)$$

Equation (3.3) possesses the same singularities in the domain D as does Eq. (1.2) in the domain Q; it degenerates for $\eta = 1$, $\eta = 0$ and $\xi = 0$, i.e. on the whole fundamental boundary of the domain D.

Existence and uniqueness theorems for the solution W of the problem (3.1), (3.2) were proved in [3]. A study of Eq. (3.3) on the basis of the properties of the solution W of the problem (3.1), (3.2) leads to the same results as those above by virtue of the invertibility (proved in [3]) of the Crocco transforamation.

We note, in conclusion, that if the integral (1.6), in which

$$\beta(y) = \min_{0 \leqslant x \leqslant X} v(x, y)$$

diverges, and the coefficients and the right side of Eq. (1.2) satisfy the Conditions A as before, then the Eq. (1.2) has a unique bounded solution in Q, satisfying only the condition (1.5).

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